UNIT #03

REDUCTION FORMULA

Obtain reduction formula for $I_n = \int \sin^n x\, dx$ and $J_n = \int_0^{\pi/2} \sin^n x\, dx$

$$I_n = \int \sin^n x\, dx$$

$$= \int \sin^{n-1} x \cdot \sin x\, dx$$

$$= \sin^{n-1} x \int \sin x\, dx - \int \left(d(\sin^{n-1} x) \int \sin x\, dx \right) dx$$

$$= \sin^{n-1} x\,(-\cos x) - \int (n-1) \sin^{n-2} x\,(\cos x)(-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,(1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int [\sin^{n-2} x - \sin^n x]\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) I_n$$

$$(\cancel{1} + n - \cancel{1}) I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$$

$$n I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx \quad \text{——— (1)}$$

Now, we will evaluate. $\int_0^{\pi/2} \sin^n x\, dx$.

$$J_n = \int_0^{\pi/2} \sin^n x\, dx$$

$$= \left[\left(-\frac{\sin^{n-1} x \cos x}{n} \right)_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx \right] \quad [\text{using eq (1)}]$$

$$= \left[0 + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx \right] = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$J_n = \frac{n-1}{n} J_{n-2}$$

$$J_{n-2} = \frac{n-3}{n-2} J_{n-4}$$

$$J_{n-4} = \frac{n-5}{n-4} J_{n-6}$$

$$\vdots$$

$$J_3 = \frac{2}{3} J_1$$

$$J_2 = \frac{1}{2} J_0$$

$$J_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{2}{3} J_1, & \text{when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{1}{2} J_0, & \text{when } n \text{ is even.} \end{cases}$$

$$J_1 = \int_0^{\pi/2} \sin x \, dx$$
$$= [-\cos x]_0^{\pi/2} = 1$$
$$J_0 = \int_0^{\pi/2} \sin^0 x \, dx$$
$$= \int_0^{\pi/2} dx = \pi/2$$

$$J_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{1}{2}\frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases} \quad \text{——— (2)}$$

Now, we obtain reduction formula for

$$I_n = \int \cos^n x \, dx \text{ and } J_n = \int_0^{\pi/2} \cos^n x \, dx.$$

$$I_n = \int \cos^{n-1} x \cos x \, dx$$

Integrating by parts, we get

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad \text{——— (3)}$$

and

$$J_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{when } n \text{ is even.} \end{cases}$$

Example (1) Evaluate $\int \cos^7 x \, dx$

Solution

$$I_7 = \int \cos^7 x \, dx$$
$$= \frac{\cos^6 x \sin x}{7} + \frac{6}{7} \int \cos^5 x \, dx \quad \text{——— (1)}$$
$$\int \cos^5 x \, dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5} \int \cos^3 x \, dx \quad \text{——— (2)}$$
$$\int \cos^3 x \, dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx$$
$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \sin x. \quad \text{——— (3)}$$
$$\therefore \int \cos^5 x \, dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5} \left[\frac{\cos^2 x \sin x}{3} + \frac{2}{3} \sin x \right] \quad [\text{From eqs (2) and (3)}]$$
$$= \frac{\cos^4 x \sin x}{5} + \frac{4 \cos^2 x \sin x}{15} + \frac{8}{15} \sin x \quad \text{——— (4)}$$

From eqs (1) and (4)

$$I_7 = \frac{\cos^6 x \sin x}{7} + \frac{6}{7} \left[\frac{\cos^4 x \sin x}{5} + \frac{4 \cos^2 x \sin x}{15} + \frac{8}{15} \sin x \right]$$
$$= \frac{\cos^6 x \sin x}{7} + \frac{6 \cos^4 x \sin x}{35} + \frac{24 \cos^2 x \sin x}{105} + \frac{48}{105} \sin x$$
$$= \frac{\cos^6 x \sin x}{7} + \frac{6 \cos^4 x \sin x}{35} + \frac{8 \cos^2 x \sin x}{35} + \frac{16 \sin x}{35} \quad \underline{\text{Ans}}$$

Example 02 Evaluate $\int_0^{\pi/2} \sin^{10} x \, dx$

Solution

$$\int_0^{\pi/2} \sin^{10} x \, dx = \frac{10-1}{10} \cdot \frac{10-3}{10-2} \cdot \frac{10-5}{10-4} \cdot \frac{10-7}{10-6} \cdot \frac{10-9}{10-8} \cdot \frac{\pi}{2}$$

$$= \frac{9}{\cancel{10}_2} \cdot \frac{7}{8} \cdot \frac{\cancel{5}}{\cancel{6}_2} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{63\pi}{512} \text{ Ans}$$

Example 03 Evaluate $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} d\theta$

Solution $I = \int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2} = \int_0^{\pi} \frac{(2\sin\theta/_2 \cos\theta/_2)^4}{(2\cos^2\theta/_2)^2} d\theta$

$$= \int_0^{\pi} \frac{2^4 \sin^4\theta/_2 \cancel{\cos^4\theta/_2}}{2^2 \cancel{\cos^4\theta/_2}} d\theta$$

$$= 4\int_0^{\pi} \sin^4\theta/_2 \, d\theta \quad \text{—— (1)}$$

Suppose $\theta/_2 = t \Rightarrow \theta = 2t$ and $d\theta = 2dt$ | At $\theta = 0$, $t = 0$; $\theta = \pi$, $t = \pi/2$.

Hence from eq(1), we get

$$I = 4\int_0^{\pi/2} \sin^4 t \, 2dt$$

$$= 8\int_0^{\pi/2} \sin^4 t \, dt = 8 \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$$

$$= \cancel{8} \cdot \frac{3}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2} = \frac{3\pi}{2} \text{ Ans}$$

Example 04 Evaluate $\int_0^1 x^5 \sqrt{\frac{1+x^2}{1-x^2}} dx$

Solution: Here $I = \int_0^1 x^5 \sqrt{\frac{1+x^2}{1-x^2}} dx$

Suppose $x^2 = \cos\theta$

$$2x\,dx = -\sin\theta \, d\theta$$

$$\Rightarrow x\,dx = -\frac{1}{2}\sin\theta \, d\theta$$

At $x = 0$, $\cos\theta = 0 \Rightarrow \theta = \pi/2$

$x = 1$, $\cos\theta = 1 \Rightarrow \theta = 0$

$$I = \int_{\pi/2}^{0} \cos^2\theta \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \left(-\frac{1}{2}\sin\theta\right) d\theta$$

$$= -\frac{1}{2}\int_{\pi/2}^{0} \cos^2\theta \sqrt{\frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin^2\theta/2}} \sin\theta \, d\theta$$

$$= \frac{1}{\cancel{2}}\int_{0}^{\pi/2} \cos^2\theta \, \frac{\cos\theta/2}{\cancel{\sin\theta/2}} \cdot \cancel{2}\cancel{\sin\theta/2}\cos\theta/2 \, d\theta$$

$$= \int_{0}^{\pi/2} \cos^2\theta \, \frac{2\cos^2\theta/2}{2} \, d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} \cos^2\theta (1+\cos\theta) \, d\theta$$

$$= \frac{1}{2}\left[\int_{0}^{\pi/2} \cos^2\theta \, d\theta + \int_{0}^{\pi/2} \cos^3\theta \, d\theta\right]$$

$$= \frac{1}{2}\left[\frac{2-1}{2}\cdot\frac{\pi}{2} + \frac{3-1}{3}\right] = \frac{1}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2}{3}\right]$$

$$= \frac{\pi}{8} + \frac{1}{3} = \frac{3\pi+8}{24} \text{ Ans}$$

Now. we obtain reduction formula for.

$$I_{p,q} = \int \sin^p x \cos^q x \, dx \text{ and}$$

$$J_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x \, dx.$$

$$I_{p,q} = \int \sin^p x \cos^q x \, dx$$

$$= \int \sin^{p-1}x \cdot \sin x \cos^q x \, dx$$

$$= \int \sin^{p-1}x \, (\sin x \cos^q x) \, dx$$

$$= \sin^{p-1}x \int \sin x \cos^q x \, dx - \int\left(\frac{d}{dx}(\sin^{p-1}x)\int \sin x \cos^q x \, dx\right) dx$$

$$= \sin^{p-1}x\left(-\frac{\cos^{q+1}x}{q+1}\right) - \int (p-1)\sin^{p-2}x \cos x \left(-\frac{\cos^{q+1}x}{q+1}\right) dx$$

$$= -\frac{\sin^{p-1}x \cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x \cos x \cos^{q+1}x \, dx.$$

$$I_{p,q} = -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q+2}x\,dx$$

$$= -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q}x\,(1-\sin^2 x)\,dx$$

$$= -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q}x\,dx - \frac{p-1}{q+1}\int \sin^{p}x\cos^{q}x\,dx$$

$$= -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q}x\,dx - \frac{p-1}{q+1}I_{p,q}$$

$$\left(1+\frac{p-1}{q+1}\right)I_{p,q} = -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q}x\,dx$$

$$\left(\frac{q+\cancel{1}+p-\cancel{1}}{q+1}\right)I_{p,q} = -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q}x\,dx$$

$$\left(\frac{p+q}{q+1}\right)I_{p,q} = -\frac{\sin^{p-1}x\cos^{q+1}x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2}x\cos^{q}x\,dx$$

$$\boxed{I_{p,q} = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}x\cos^{q}x\,dx} \quad \text{——} (1)$$

Now, we will find $J_{p,q} = \int_0^{\pi/2} \sin^p x\cos^q x\,dx$, where p and q are positive integers.

From eq (1), we get

$$J_{p,q} = -\left[\frac{\sin^{p-1}x\cos^{q+1}x}{p+q}\right]_0^{\pi/2} + \frac{p-1}{p+q}\int_0^{\pi/2}\sin^{p-2}x\cos^{q}x\,dx$$

$$= \frac{p-1}{p+q}\int_0^{\pi/2}\sin^{p-2}x\cos^{q}x\,dx$$

$$= \frac{p-1}{p+q}J_{p-2,q}.$$

$$J_{p-2,q} = \frac{p-3}{p+q-2}J_{p-4,q}.$$

$$J_{p-4,q} = \frac{p-5}{p+q-4}J_{p-6,q}.$$

$$J_{3,q} = \frac{2}{q+3} J_{1,q}$$

$$J_{2,q} = \frac{1}{q+2} J_{0,q}$$

$$J_{p,q} = \begin{cases} \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \frac{p-5}{p+q-4} \cdots \frac{2}{q+3} J_{1,q}, & \text{if } p \text{ is odd} \\ \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \frac{p-5}{p+q-4} \cdots \frac{1}{q+2} J_{0,q}, & \text{if } p \text{ is even} \end{cases} \quad \text{—— (2)}$$

$$J_{1,q} = \int_0^{\pi/2} \sin x \cos^q x \, dx$$

$$= -\left[\frac{\cos^{q+1} x}{q+1}\right]_0^{\pi/2}$$

$$= -\left[-\frac{1}{q+1}\right] = \frac{1}{q+1}$$

Now, $J_{0,q} = \int_0^{\pi/2} \sin^0 x \cos^q x \, dx$

$$= \int_0^{\pi/2} \cos^q x \, dx$$

$$= \begin{cases} \frac{q-1}{q} \cdot \frac{q-3}{q-2} \cdots \frac{2}{3}, & \text{when } q \text{ is odd} \\ \frac{q-1}{q} \cdot \frac{q-3}{q-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{when } q \text{ is even.} \end{cases}$$

From eq (2), we get

$$J_{p,q} = \begin{cases} \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdots \frac{2}{q+3} \cdot \frac{1}{q+1}, & \text{when } p \text{ is odd} \\ \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdots \frac{1}{q+2} \cdot \frac{q-1}{q} \cdot \frac{q-3}{q-2} \cdots \frac{2}{3}, & \text{when } p \text{ is even and } q \text{ is odd.} \\ \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdots \frac{1}{q+2} \cdot \frac{q-1}{q} \cdot \frac{q-3}{q-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{when } p \text{ is even and } q \text{ is even.} \end{cases}$$

$$J_{p,q} = \begin{cases} \frac{(p-1)(p-3)\cdots(q-1)(q-3)}{(p+q)(p+q-2)\cdots} \cdot \frac{\pi}{2}, & \text{when both } p \text{ and } q \text{ are even} \\ \frac{(p-1)(p-3)\cdots(q-1)(q-3)}{(p+q)(p+q-2)\cdots}, & \text{otherwise} \end{cases}$$

$$I_{p,q} = \int \sin^p x \cos^q x \, dx$$

$$= \int \cos^{q-1} x \, (\cos x \sin^p x) \, dx$$

$$= \cos^{q-1} x \int \cos x \sin^p x \, dx - \int \left(d(\cos^{q-1} x) \int \cos x \sin^p x \, dx \right) dx$$

$$\boxed{I_{p,q} = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} I_{p,q-2}.}$$

Example: Evaluate $\int \frac{\cos^5 x}{\sin x} dx$

Solution: Suppose $\sin x = t$

then $\cos x \, dx = dt$

$$\therefore \int \frac{\cos^5 x}{\sin x} dx = \int \frac{\cos^4 x}{t} dt$$

$$= \int \frac{(\cos^2 x)^2}{t} dt$$

$$= \int \frac{(1-\sin^2 x)^2}{t} dt$$

$$= \int \frac{(1-t^2)^2}{t} dt$$

$$= \int \frac{(1+t^4-2t^2)}{t} dt$$

$$= \int \left(\frac{1}{t} + t^3 - 2t \right) dt$$

$$= \log t + \frac{t^4}{4} - \cancel{2}\frac{t^2}{\cancel{2}} + c$$

$$= \log(\sin x) + \frac{1}{4}(\sin x)^4 - \sin^2 x + c$$

$$= \log(\sin x) + \frac{\sin^4 x}{4} - \sin^2 x + c$$

Ans

Example. Evaluate $\int \sin^2 x \cos^6 x \, dx$

Solution

$$I_{2,6} = \int \sin^2 x \cos^6 x \, dx$$

$$= \frac{\sin^3 x \cos^5 x}{8} + \frac{5}{8}\int \sin^2 x \cos^4 x \, dx \quad \text{---(1)}$$

$$\int \sin^2 x \cos^4 x \, dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x \, dx \quad \text{---(2)}$$

$$\int \sin^2 x \cos^2 x \, dx = \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \sin^2 x \cos^0 x \, dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \sin^2 x \, dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{4}\int \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{8}\int (1-\cos 2x) \, dx$$

$$= \frac{\sin^3 x \cos x}{4} + \frac{1}{8}\left(x - \frac{\sin 2x}{2}\right) \quad \text{---(3)}$$

Now, from eqs (2) & (3), we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\left[\frac{\sin^3 x \cos x}{4} + \frac{1}{8}\left(x - \frac{\sin 2x}{2}\right)\right]$$

$$= \frac{\sin^3 x \cos^3 x}{6} + \frac{\cancel{3}\sin^3 x \cos x}{\cancel{24}\;8} + \frac{\cancel{3}x}{\cancel{48}\;16} - \frac{\cancel{3}\sin 2x}{\cancel{96}\;32}$$

$$= \frac{\sin^3 x \cos^3 x}{6} + \frac{\sin^3 x \cos x}{8} + \frac{x}{16} - \frac{\sin 2x}{32} \quad \text{---(4)}$$

From eqs (1) and (4),

$$I_{2,6} = \frac{\sin^3 x \cos^5 x}{8} + \frac{5}{8}\left[\frac{\sin^3 x \cos^3 x}{6} + \frac{\sin^3 x \cos x}{8} + \frac{x}{16} - \frac{\sin 2x}{32}\right]$$

$$= \frac{\sin^3 x \cos^5 x}{8} + \frac{5\sin^3 x \cos^3 x}{48} + \frac{5\sin^3 x \cos x}{64} + \frac{5(x - \sin x \cos x)}{128}$$

Ans.

Example: Evaluate $\int_0^{\pi/4} \cos^3 2x \sin^4 4x \, dx$

Solution: Suppose $2x = \theta$, At $x = 0, \theta = 0$

$2dx = d\theta$ $\qquad x = \pi/4, \theta = \frac{\pi}{2}$

$$\int_0^{\pi/4} \cos^3 2x \sin^4 4x \, dx = \int_0^{\pi/2} \cos^3\theta \sin^4 2\theta \frac{d\theta}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^3\theta \, (\sin 2\theta)^4 d\theta$$

$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^3\theta \, (2\sin\theta\cos\theta)^4 d\theta$$

$$= \frac{2^4}{2}\int_0^{\pi/2} \cos^3\theta \sin^4\theta \cos^4\theta \, d\theta$$

$$= 8\int_0^{\pi/2} \cos^7\theta \sin^4\theta \, d\theta$$

$\int_0^{\pi/2} \sin^p x \cos^q x \, dx$
$= \frac{(p-1)(p-3)\cdots(q-1)(q-3)\cdots}{(p+q)(p+q-2)\cdots}$
multiply by $\frac{\pi}{2}$ if p and q both are even

$$= 8\,\frac{(7-1)(7-3)(7-5)(4-1)(4-3)}{(7+4)(7+4-2)(7+4-4)(7+4-6)(7+4-8)}$$

$$= \frac{8\cdot \cancel{6}^{\,2}\cdot \cancel{4}\cdot 2\cdot \cancel{3}\cdot 1}{11\cdot \cancel{9}_{\,3}\cdot 7\cdot 5\cdot \cancel{3}}$$

$$= \frac{128}{1155} \quad \text{Ans}$$

Example: Evaluate $\int_0^2 x^3\sqrt{2x - x^2}\,dx$

Solution: Suppose $x = 2\sin^2\theta$. $\quad$ At $x = 0, \theta = 0$

$dx = 4\sin\theta\cos\theta\,d\theta \quad x = 2, \theta = \frac{\pi}{2}$

$$\int_0^2 x^3\sqrt{2x - x^2}\,dx = \int_0^{\pi/2} (2\sin^2\theta)^3 \sqrt{2.2\sin^2\theta - 4\sin^4\theta}\,.\,4\sin\theta\cos\theta\,d\theta.$$

$$= \int_0^{\pi/2} 8\sin^6\theta \,.\, 2\sin\theta\sqrt{1 - \sin^2\theta}\,.\,4\sin\theta\cos\theta\,d\theta$$

$$= 64\int_0^{\pi/2} \sin^8\theta \,.\, \cos^2\theta\,d\theta$$

$$= 64.\frac{(8-1)(8-3)(8-5)(8-7).(2-1)}{(8+2)(8+2-2)(8+2-4)(8+2-6)(8+2-8)}\cdot\frac{\pi}{2}$$

$$= \frac{\cancel{64}\cdot 7\cdot \cancel{5}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{10}_{2}\cdot \cancel{8}_{2}\cdot \cancel{6}\cdot \cancel{4}\cdot \cancel{2}}\cdot\frac{\pi}{2} = \frac{7\pi}{8} \quad \text{Ans}.$$

03/09

~~Example~~: obtain reduction formula for

$$I_n = \int \tan^n x\,dx \text{ and } J_n = \int_0^{\pi/4} \tan^n x\,dx, \quad n \in N$$

$$I_n = \int \tan^n x\,dx$$
$$= \int \tan^{n-2} x \tan^2 x\,dx.$$
$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$
$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$\boxed{I_n = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx}$$

$$J_n = \int_0^{\pi/4} \tan^n x\,dx$$
$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x\,dx.$$

$$\boxed{J_n = \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2} x\,dx}$$

Similarly, we can find

$$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx.$$

and $$\int_{\pi/4}^{\pi/2} \cot^n x\,dx = \frac{1}{n-1} - \int_{\pi/4}^{\pi/2} \cot^{n-2} x\,dx.$$

Example: Evaluate $\int \tan^6 x\,dx$

Solution: $$\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \int \tan^4 x\,dx. \quad \text{—— (1)}$$

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx \quad \text{—— (2)}$$

$$\int \tan^2 x\,dx = \tan x - \int \tan^0 x\,dx$$
$$= \tan x - x \quad \text{—— (3)}$$

From eqs (2) and (3)

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - (\tan x - x)$$
$$= \frac{\tan^3 x}{3} - \tan x + x \quad \text{—— (4)}$$

From eqs (1) and (4)

$$\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - \tan x + x\right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x. \quad \underline{\text{Ans}}$$

Example: Evaluate $\int_{\pi/4}^{\pi/2} \cot^4 x\,dx$

Solution:

$$\int_{\pi/4}^{\pi/2} \cot^4 x\,dx = \frac{1}{3} - \int_{\pi/4}^{\pi/2} \cot^2 x\,dx \quad —(1)$$

$$\int_{\pi/4}^{\pi/2} \cot^2 x\,dx = 1 - \int_{\pi/4}^{\pi/2} \cot^0 x\,dx$$

$$= 1 - [x]_{\pi/4}^{\pi/2}$$

$$= 1 - \left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= 1 - \frac{\pi}{4} \quad —(2)$$

From eqs (1) and (2)

$$\int_{\pi/4}^{\pi/2} \cot^4 x\,dx = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3} = \frac{3\pi - 8}{12} \quad \underline{\text{Ans.}}$$

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